NOTE ON THE CHIEF THEOREM OF LIE'S THEORY OF CONTINUOUS GROUPS.

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The chief (*haupt*) theorem of Lie's theory of finite continuous groups is that a system of r independent infinitesimal transformations

$$X_1, X_2, \ldots X_r,$$

such that

$$(X_j, X_k) = \sum_1^r{}_s\, c_{jks}\, X_s^* \qquad (j, k = 1, 2 \ldots r),$$

for constant coefficients c_{jks}, generates a continuous group with r parameters, that is, a group with r parameters in which each transformation can be generated by an infinitesimal transformation of the group.†

Professor Study, however, has shown that, notwithstanding the infinitesimal transformations of the special linear homogeneous group satisfy Lie's criterion, nevertheless not every transformation can be generated by an infinitesimal transformation of this group.‡ Consequently Lie's theorem is subject to certain limitations. The precise nature of the error in the demonstration of Lie's theorem has, so far as I know, not been pointed out; and to show wherein it consists is the object of this paper. For this purpose I shall carry out, for the case of a particular group, the successive steps (as given in the "Continuierliche Gruppen," pp. 368–377) in Lie's demonstration of the first fundamental theorem of his theory, upon which the chief theorem (the *second fundamental theorem*) depends. At a certain point in this demonstration an assumption is made in which Lie's error consists.

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* X_s denotes $\xi_{s1}(x_1 \ldots x_n)\dfrac{\partial}{\partial x_1} + \ldots + \xi_{sn}(x_1 \ldots x_n)\dfrac{\partial}{\partial x_n}$.

† Lie: Continuierliche Gruppen, pp. 211, 390.

‡ Engel: Leipziger Berichte, 1892, p. 279. See also Taber: Am. Jour. Maths., XVI.; Bull. N. Y. Math. Soc., July, 1894, April, 1896, Jan., 1897; Math. Ann., XLVI.; Rettger: Proc. Am. Acad., XXXIII.; Newson: Kansas Univ. Quarterly, 1896; and These Proceedings, p. 97.

I shall number the equations the same, and shall use the same notation for the special group considered as employed by Lie in his general demonstration.

Lie first shows that if the n equations

$$(1) \qquad x_i' = f_i(x_1, \ldots x_n, \; a_1 \ldots a_r) \quad (i = 1, 2, \ldots n)$$

represent a group with r parameters, the x''s as functions of the x's and a's satisfy certain differential equations of the form

$$(9) \qquad \frac{\partial x_i'}{\partial a_k} = \sum_{1}^{r}{}_j \, \psi_{jk}(a_1 \ldots a_r)\, \xi_{ji}(x_1' \ldots x_n')$$
$$(i = 1, 2, \ldots n; \; k = 1, 2, \ldots r),$$

in which the determinant of the $\psi_{jk} \mp 0$; — that, consequently, these equations may be written in the form

$$(10) \qquad \xi_{ji}(x_1' \ldots x_n') = \sum_{1}^{r}{}_k \, \alpha_{jk}(a_1 \ldots a_r) \frac{\partial x_i'}{\partial a_k}$$
$$(i = 1, 2, \ldots n; \; j = 1, 2, \ldots r),$$

where the determinant of the $\alpha_{jk} \mp 0$; and that, further, no linear relation of the form

$$e_1 \xi_{1i}(x') + \ldots + e_r \xi_{ri}(x') \equiv 0,$$

with constant coefficients e, persists, simultaneously, for $i = 1, 2, \ldots n$.

We shall consider a case for which both n and r are equal to two. The equations

$$(1) \qquad \begin{aligned} x_1' &= x_1 + a_2 && \equiv f_1(x, a), \\ x_2' &= e^{a_2} x_2 + a_1 && \equiv f_2(x, a), \end{aligned}$$

define ∞^2 of transformations T_a which constitute a group. For, by the elimination of x_1', x_2' from (1) and

$$(2) \qquad \begin{aligned} x_1'' &= x_1' + b_2 && \equiv f_1(x', b), \\ x_2'' &= e^{b_2} x_2' + b_1 && \equiv f_2(x', b), \end{aligned}$$

we derive

$$(3) \qquad \begin{aligned} x_1'' &= x_1 + c_2 && \equiv f_1(x, c), \\ x_2'' &= e^{c_2} x_2 + c_1 && \equiv f_2(x, c), \end{aligned}$$

where

$$(4) \qquad \begin{aligned} c_1 &= a_1 e^{b_2} + b_1 && \equiv \phi_1(a, b), \\ c_2 &= a_2 + b_2 && \equiv \phi_2(a, b). \end{aligned}$$

That is, functional equations persist of the form

$$(5) \qquad f_i(f(x, a), b) = f_i(x, \phi(a, b)) \quad (i = 1, 2).$$

Therefore, the composition of two transformations T_a and T_b of the family is equivalent to a single transformation of the family. It is to be observed, as noted by Lie, § 1 in the demonstration of the general case, that the functions ϕ_1 and ϕ_2 are independent of each other with respect to b_1 and b_2. For

$$\left| \frac{\partial \phi_h}{\partial b_j} \right| \equiv \left| \begin{matrix} 1, & a_1 e^{b_2} \\ 0, & 1 \end{matrix} \right|$$

is not identically zero.

We may, therefore, regard $x_1, x_2, a_1, a_2, c_1, c_2$, as independent variables, and $x_1', x_2', x_1'', x_2'', b_1, b_2$, as dependent variables. Then the differentiation of the functional equations (5), or of

$$(5') \qquad f_\lambda(x', b) = f_\lambda(x, c) \quad (\lambda = 1, 2),$$

i. e., of

$$(5'a) \qquad \begin{aligned} x_1' + b_2 &= x_1 + c_2, \\ b_1 + e^{b_2} x_2' &= c_1 + e^{c_2} x_2, \end{aligned}$$

with respect to the a's gives

$$(7) \qquad \begin{aligned} &\frac{\partial x_1'}{\partial a_1} + \frac{\partial b_2}{\partial a_1} = 0, \\ &\frac{\partial x_1'}{\partial a_2} + \frac{\partial b_2}{\partial a_2} = 0, \\ &e^{b_2} \frac{\partial x_2'}{\partial a_1} + \frac{\partial b_1}{\partial a_1} + x_2' e^{b_2} \frac{\partial b_2}{\partial a_1} = 0, \\ &e^{b_2} \frac{\partial x_2'}{\partial a_2} + \frac{\partial b_1}{\partial a_2} + x_2' e^{b_2} \frac{\partial b_2}{\partial a_2} = 0. \end{aligned}$$

In order to obtain expressions for $\dfrac{\partial b_j}{\partial a_k}$, we differentiate (4) with respect to a_1 and a_2, and thus obtain

$$0 = \frac{\partial b_2}{\partial a_1},$$

$$0 = 1 + \frac{\partial b_2}{\partial a_2},$$

$$0 = e^{b_2} + \frac{\partial b_1}{\partial a_1} + a_1 e^{b_2} \frac{\partial b_2}{\partial a_1},$$

$$0 = \frac{\partial b_1}{\partial a_2} + a_1 e^{b_2} \frac{\partial b_2}{\partial a_2}.$$

These equations give

$$\frac{\partial b_1}{\partial a_1} = -e^{b_2}, \qquad \frac{\partial b_1}{\partial a_2} = a_1 e^{b_2}, \qquad \frac{\partial b_2}{\partial a_1} = 0, \qquad \frac{\partial b_2}{\partial a_2} = -1;$$

and, inserting these values, equations (7) become

$$(8\,a) \qquad \begin{aligned} \frac{\partial x_1'}{\partial a_1} &= 0, \\ \frac{\partial x_1'}{\partial a_2} &= 1, \\ e^{b_2} \frac{\partial x_2'}{\partial a_1} &= e^{b_2}, \\ e^{b_2} \frac{\partial x_2'}{\partial a_2} &= e^{b_2} (x_2' - a_1). \end{aligned}$$

If in $(8\,a)$ we insert the values of x_1' and x_2' derived from

$$(1) \qquad \begin{aligned} x_1' &= x_1 + a_2, \\ x_2' &= e^{a_2} x_2 + a_1, \end{aligned}$$

they become equations between the independent quantities x_1, x_2, a_1, a_2, b_1, b_2; and must, therefore, be satisfied identically. Hence, equations $(8\,a)$ will still persist identically, in virtue of equations (1), if we assign definite values to the b's.

For this purpose let $e^{b_2} = 1$. We then have

$$(9\,a) \quad \frac{\partial x_1'}{\partial a_1} = 0, \qquad \frac{\partial x_1'}{\partial a_2} = 1, \qquad \frac{\partial x_2'}{\partial a_1} = 1, \qquad \frac{\partial x_2'}{\partial a_2} = x_2' - a_1.$$

Therefore, if we define functions ξ of the x''s and functions ψ of the a's as follows,

$$\xi_{11}(x') \equiv 0, \quad \xi_{12}(x') \equiv -1, \quad \xi_{21}(x') \equiv -1, \quad \xi_{22}(x') \equiv -x_2',$$

$$\psi_{11}(a) \equiv -1, \quad \psi_{12}(a) \equiv a_1, \quad \psi_{21}(a) \equiv 0, \quad \psi_{22}(a) \equiv -1,$$

we have

$$(9) \qquad \frac{\partial x_i'}{\partial a_k} = \psi_{1k}(a)\,\xi_{1i}(x') + \psi_{2k}(a)\,\xi_{2i}(x') \qquad (i = 1, 2;\ k = 1, 2).$$

The determinant of the ψ's, namely,

$$\begin{vmatrix} \psi_{11}, & \psi_{12} \\ \psi_{21}, & \psi_{22} \end{vmatrix} = \begin{vmatrix} -1, & a_1 \\ 0, & -1 \end{vmatrix},$$

not being identically zero, equations (9) may be solved for the ξ's, giving,

$$(10) \qquad \xi_{ji}(x_1', x_2') = \alpha_{j1}(a)\frac{\partial x_i'}{\partial a_1} + \alpha_{j2}(a)\frac{\partial x_i'}{\partial a_2} \qquad (i = 1, 2;\ j = 1, 2),$$

where

$$\alpha_{11}(a_1, a_2) \equiv -1, \qquad \alpha_{12}(a_1, a_2) \equiv 0,$$

$$\alpha_{21}(a_1, a_2) \equiv -a_1, \qquad \alpha_{22}(a_1, a_2) \equiv -1.$$

It is to be observed that

$$e_1\,\xi_{11}(x') + e_2\,\xi_{21}(x') \equiv -e_2,$$

$$e_1\,\xi_{12}(x') + e_2\,\xi_{22}(x') \equiv -e_1 - e_2 x_2'.$$

Therefore, these expressions linear in the ξ's cannot both be simultaneously zero (for all values of the x''s) if e_1 and e_2 are constants other than zero. That is to say, no two constants e_1 and e_2, not zero, can be found for which

$$e_1\,\xi_{1i}(x') + e_2\,\xi_{2i}(x') \equiv 0,$$

for $i = 1$ and $i = 2$, simultaneously.

We come next to the demonstration of the second part of Lie's first fundamental theorem. Starting with a system of equations which define a family with r essential parameters ($r = 2$ in the case considered), and satisfying differential equations of the form (9), we proceed to follow the steps in Lie's demonstration that this family (provided certain other conditions are also satisfied) constitutes a group.

As shown above, the family of ∞^2 of transformations T_a defined by

$$(1) \qquad \begin{aligned} x_1' &= x_1 + a_2 \equiv f_1(x, a), \\ x_2' &= e^{a_2} x + a_1 \equiv f_2(x, a), \end{aligned}$$

satisfy differential equations of the form (9) in which the determinant of the ψ_{jk} is not identically zero; also the differential equations

$$\xi_{ji}(x_1', x_2') = a_{j1}(a)\frac{\partial x_i'}{\partial a_1} + a_{j2}(a)\frac{\partial x_i'}{\partial a_2}$$
$$(i = 1, 2; j = 1, 2),$$

where the ξ's and α's are defined as follows:

$$\xi_{11}(x') \equiv 0, \quad \xi_{12}(x') \equiv -1, \quad \xi_{21}(x') \equiv -1, \quad \xi_{22}(x') \equiv -x_2',$$
$$\alpha_{11}(a) \equiv -1, \quad \alpha_{12}(a) \equiv 0, \quad \alpha_{21}(a) \equiv -a_1, \quad \alpha_{22}(a) \equiv -1.$$

Moreover, if we put $a_1^{(0)} = a_2^{(0)} = 0$, then $a_1 = a_1^{(0)}$, $a_2 = a_2^{(0)}$ gives the identical transformation; and the determinant of the $\alpha_{jk}(a^{(0)})$, namely,

$$\begin{vmatrix} \alpha_{11}^{(0)}, & \alpha_{12}^{(0)} \\ \alpha_{21}^{(0)}, & \alpha_{22}^{(0)} \end{vmatrix} = \begin{vmatrix} -1, & 0 \\ -a, & -1 \end{vmatrix}$$

is neither zero nor infinite.

In order to prove that this family constitutes a group, we proceed to integrate equations (10). For this purpose, introduce a new auxiliary variable t by means of the equations

$$(11) \quad \begin{aligned} \frac{da_1}{dt} &= \lambda_1 \alpha_{11}(a_1, a_2) + \lambda_2 \alpha_{21}(a_1, a_2) \equiv -\lambda_1 - a_1 \lambda_2, \\ \frac{da_2}{dt} &= \lambda_1 \alpha_{12}(a_1, a_2) + \lambda_2 \alpha_{22}(a_1, a_2) \equiv -\lambda_2, \end{aligned}$$

where λ_1 and λ_2 are any arbitrary but definite constants. To determine the constants of integration, we assume that a_1, a_2 take the values $\bar{a}_1$, $\bar{a}_2$ for $t = \bar{t}$. The integrals of equations (11) are then

$$a_2 = \bar{a}_2 - \lambda_2(t - \bar{t}),$$

$$\log\left[\frac{(\lambda_1 + a_1\lambda_2)}{(\lambda_1 + \bar{a}_1\lambda_2)}\right] = -\lambda_2(t - \bar{t})$$

Let now

$$\lambda_1(t - \bar{t}) = \mu_1, \quad \lambda_2(t - \bar{t}) = \mu_2;$$

the integral equations then become

$$(12) \quad \begin{aligned} a_1 &= \frac{\mu_1}{\mu_2 e^{\mu_2}} + \frac{\bar{a}_1}{e^{\mu_2}} - \frac{\mu_1}{\mu_2} \equiv \Phi_1(\mu, \bar{a}), \\ a_2 &= \bar{a}_2 - \mu_2 \equiv \Phi_2(\mu, \bar{a}). \end{aligned}$$

It is to be observed that the a's are independent functions of the μ's: for

$$\left| \frac{\partial a_k}{\partial \mu_j} \right| \equiv \frac{1}{\mu_2}\left(\frac{e^{\mu_2}-1}{e^{\mu_2}}\right)$$

is not identically zero.

By means of equations (12) we can introduce in equations (1) the new parameters μ_1, μ_2 instead of a_1, a_2. Solving (12) for μ_1, μ_2, we have

$$(13) \qquad \begin{aligned} \mu_1 &= \frac{(\bar{a}_2 - a_2)\left(a_1 - \dfrac{\bar{a}_1}{e^{\bar{a}_2 - a_2}}\right) e^{\bar{a}_2 - a_2}}{1 - e^{\bar{a}_2 - a_2}} \equiv \mathrm{M}_1(a, \bar{a}), \\ \mu_2 &= \bar{a}_2 - a_2 \equiv \mathrm{M}_2(a, \bar{a}). \end{aligned}$$

If we introduce these new parameters μ_1, μ_2 in equations (1), the x''s become definite functions of x_1, x_2, μ_1, μ_2, $\bar{a}_1$, $\bar{a}_2$; and, since the μ's contain t, the x''s will also be functions of the auxiliary variable t. Multiplying equations (10) by λ_1, λ_2, and adding, we get

$$\sum_1^2{}_j \lambda_j \xi_{ji}(x_1' \; x_2') = \sum_1^2{}_k \frac{\partial x_i'}{\partial a_k}\frac{d a_k}{dt} \qquad (i = 1, 2),$$

or

$$(14) \qquad \begin{aligned} \frac{dx'_1}{dt} &= \lambda_1 \xi_{11}(x_1', x_2') + \lambda_2 \xi_{21}(x_1', x_2') \equiv -\lambda_2, \\ \frac{dx_2'}{dt} &= \lambda_1 \xi_{12}(x_1', x_2') + \lambda_2 \xi_{22}(x_1', x_2') \equiv -\lambda_1 - \lambda_2 x_2', \end{aligned}$$

which define the x''s as functions of t. Since a_1, a_2 reduce to $\bar{a}_1$, $\bar{a}_2$ for $t = \bar{t}$, we see that x_1', x_2', for $t = \bar{t}$, take the initial values

$$(15) \qquad \begin{aligned} \bar{x}_1' &= x_1 + \bar{a}_2 \equiv f_1(x, \bar{a}), \\ \bar{x}_2' &= e^{\bar{a}_2} x_2 + \bar{a}_1 \equiv f_2(x, \bar{a}). \end{aligned}$$

Further, the integrals of equations (14) contain the λ's only in the combination μ_1, μ_2; and, therefore, can be expressed in the form

$$(16) \qquad \begin{aligned} x_1' &= F_1(\bar{x}_1', \bar{x}_2', \mu_1, \mu_2) \equiv \bar{x}_1' - \mu_2, \\ x_2' &= F_2(\bar{x}_1', \bar{x}_2', \mu_1, \mu_2) \equiv \frac{\mu_1}{\mu_2}\left(\frac{1}{e^{\mu_2}} - 1\right) + \frac{\bar{x}_2'}{e^{\mu_2}}. \end{aligned}$$

Inserting therein the values

$$x_i' = f_i(x, \bar{a}), \qquad x_i' = f_i(x, a), \qquad a_k = \Phi_k(\mu, \bar{a})$$
$$(i = 1, 2;\; k = 1, 2),$$

the totality of equations (1) to (12), we obtain the functional equations

$$(17) \qquad F_i(f(x, \bar{a}), \mu) = f_i(x, \Phi(\mu, \bar{a})) \qquad (i = 1, 2).$$

We have already denoted by T_a the transformation defined by (1), in which the parameters are a_1, a_2. Consequently, equations (15) define the transformation $T_{\bar{a}}$. We may now denote the transformation defined by (16) in which the parameters are μ_1, μ_2 by E_μ. Then the functional equations (17) may be expressed in the single formula

$$(18) \qquad T_{\bar{a}} E_\mu = T_a.$$

For $T_{\bar{a}}$ transforms x_i into $\bar{x}_i' = f(x, \bar{a})$, and E_μ transforms $\bar{x}_i$ into $F_i(\bar{x}', \mu)$, while, in virtue of (12), T_a can also be written in the form

$$x_i' = f_i(x, \Phi(\mu, \bar{a})) \qquad (i = 1, 2).$$

The relation (18) persists, therefore, provided the three parameter systems $(\bar{a}_1, \bar{a}_2)$, (μ_1, μ_2), (a_1, a_2) are connected by relations (12). Therein $\bar{a}_1$, $\bar{a}_2$ denote definitely chosen general values of a_1, a_2.

We now make use of the assumption that the transformation T_a, defined by (1), shall become the identical transformation for $a_1 = a_1^{(0)}$, $a_2 = a_2^{(0)}$. Namely, for the moment, let $\bar{a}_1 = a_1^{(0)}$, $\bar{a}_2 = a_2^{(0)}$; whereupon $T_{\bar{a}}$ becomes the identical transformation $T_{a^{(0)}}$. Then, because the determinant of the $a_{jk}(a^{(0)})$ is, according to assumption, neither zero nor infinite, and, therefore, the former considerations are also valid for $\bar{a} = a^{(0)}$, — the a_1, a_2 in virtue of (12) assume the values

$$(19) \qquad a_k = \Phi_k(\mu_1, \mu_2, a_1^{(0)}, a_2^{(0)}) \qquad (k = 1, 2);$$

that is to say, since we take $a_1^{(0)} = a_2^{(0)} = 0$, a_1 and a_2 assume the values

$$(19, a) \qquad a_1 = \frac{\mu_1}{\mu_2 e^{\mu_2}} - \frac{\mu_1}{\mu_2}, \qquad a_2 = -\mu_2.$$

Hence, (18) becomes

$$T_{a^{(0)}} E_\mu = T_a;$$

and, therefore,

$$(20) \qquad E_\mu = T_a.$$

Whence it follows that *each transformation* E_μ *belongs to the family of transformations* T_a, *defined by equations* (1). If now, conversely, we could establish that each transformation T_a belonged to the family of

transformations E_μ, the first fundamental theorem would then be proved. For taking the α's arbitrarily, we could then find a system of parameters μ such that $E_\mu = T_\alpha$; and, the $\bar{a}$'s being also an arbitrarily chosen system of values of the parameters a of equations (1), we should have, symbolically,

$$T_{\bar{a}}\, T_\alpha = T_{\bar{a}}\, E_\mu = T_a,$$

or

$$f_i(f(x, \bar{a}), \alpha) = F_i(f(x, a), \mu) = f_i(x, a)$$
$$(i = 1, 2),$$

where

$$a_k = \Phi_k(\mu, \bar{a}) = \Phi_k(M(\alpha, a^{(0)}), \bar{a});$$

that is to say, the composition of two arbitrary transformations $T_{\bar{a}}$ and T_α of the family gives again a transformation T_a of the family. This is precisely the step taken by Lie, who assumes that because, — as mentioned above, page 244, — the Φ_k's are independent functions of the μ's, each transformation of (1) belongs to the family E_μ. But, although the functions α_1, α_2, as defined by (19 a), are independent of the μ's, since the Jacobian

$$\frac{\partial(\alpha_1, \alpha_2)}{\partial(\mu_1, \mu_2)} \equiv \frac{1}{\mu_2}\left(1 - \frac{1}{e^{\mu_2}}\right)$$

is not identically zero, nevertheless, for certain values of the α's, the μ's are infinite. Infinite values of the μ's, however, are expressly excluded from consideration. For $\mu_k = \lambda_k(t - \bar{t})$, and since t and $\bar{t}$ cannot be infinite, if μ_k is infinite λ_k is infinite; and, by supposition, the λ's are arbitrary but definite constants in the integration on page 244. So we cannot assume that every transformation T_α of the family (1) belongs to the family E_μ. Thus, solving equations (19, a), we have

$$(19, b)\qquad \mu_1 = \frac{\alpha_1\, \alpha_2}{1 - e^{\alpha_2}}, \qquad \mu_2 = -\alpha_2.$$

For $\alpha_1 \neq 0$, and α_2 an even multiple of πi, μ_1 becomes infinite. Moreover, this transformation of the family (1) is distinct from any transformation of this family T_α for which the μ's are finite.

On page 375 of the "Continuierliche Gruppen" Lie points out that every transformation of the family E_μ is generated by an infinitesimal transformation. The infinitesimal transformation in question is represented by the symbol

$$\sum_1^r{}_j \lambda_j X_j f \equiv \sum_1^r{}_j \sum_1^n{}_i \lambda_j \xi_{ji} (x_1 \ldots x_n) \frac{\partial f}{\partial x_i},$$

and adds to an arbitrary function $f(x_1' \ldots x_n')$ the increment

$$\Sigma \lambda_j X_j' f \cdot \delta t,$$

and, therefore, to x_i adds the increment

$$\delta x_i = \xi_{ji} (x_1 \ldots x_n) \cdot \delta t.$$

This shows its relation to the simultaneous system on page 245, namely,

$$\frac{dx_i'}{dt} = \sum_1^r{}_j \lambda_j \xi_{ji} (x_1' \ldots x_n') \qquad (i = 1, 2 \ldots n).$$

If the theorem stated by Lie, page 375, "Continuierliche Gruppen," was true without exception, namely, that every transformation of the family T_a belonged to the family E_μ, it would then follow that every transformation of the family T_a could be generated by an infinitesimal transformation; for then taking the a's arbitrarily, we should have

$$E_\mu = T_a.$$

But, for a system of values of the a's for which one or both of the functions $M_1 (a, a^{(0)})$, $M_2 (a, a^{(0)})$ are infinite, there is no equivalent transformation of the family E_μ; and, consequently, such a transformation cannot be generated by an infinitesimal transformation of the group. E. g., the transformation T_a considered above, for which $a_1 \neq 0$ and a_2 is an even multiple of πi, cannot be generated by an infinitesimal transformation of the group.

In demonstrating the second fundamental theorem (the chief theorem) Lie assumes the results of the first fundamental theorem. He shows that a system of r independent infinitesimal transformations *

$$X_1 f \equiv \sum_1^n{}_k \xi_{ik} (x_1 \ldots x_n) \frac{\partial f}{\partial x_k} \qquad (i = 1, 2 \ldots r)$$

generate a family of transformations $\mathfrak{T}_a$, with r essential parameters, which contains the identical transformation, and is defined by the equations

* Lie terms the infinitesimal transformations or symbols of infinitesimal transformations $X_1, X_2, \ldots X_r$ *independent* if they satisfy no linear relation $e_1 X_1 f + \ldots + e_r X_r f \equiv 0$, with *constant* coefficients e.

$$\text{(A)} \qquad x_i' = x_i + \sum_1^r{}_k a_k X_k x_i + \tfrac{1}{2} \sum_1^r{}_k \sum_1^r{}_l a_k a_l X_k X_l x_i + \ldots \equiv \mathfrak{f}_i(x, a)$$
$$(i = 1, 2 \ldots r);$$

further, that, if and only if

$$(X_j X_k) \equiv \sum_1^r{}_s c_{jks} X_s f,$$

will this family satisfy differential equations of the form required by the first fundamental theorem. Consequently, only if this criterion is satisfied by the infinitesimal transformations can they generate a group.

Proceeding now, as in the demonstration of the first fundamental theorem, we introduce certain new parameters μ, and, finally, obtain the equation

$$\mathfrak{T}_{\bar{a}} \mathfrak{E}_\mu = \mathfrak{T}_a,$$

where $a_k = \Phi_k(\mu, \bar{a})$ $(k = 1, 2, \ldots r)$. As before, since the family of transformations $\mathfrak{T}_a$, defined now by equations (A), contains the identical transformation, we have

$$\mathfrak{E}_\mu = \mathfrak{T}_a,$$

where $a_k = \Phi_k(\mu, a^{(0)})$, and $\mu_k = \mathfrak{M}_k(a, a^{(0)})$ $(k = 1, 2, \ldots r)$,* and thus

$$\mathfrak{T}_{\bar{a}} \mathfrak{T}_a = \mathfrak{T}_{a'}.$$

In the former case we saw, page 247, that, if the a's were chosen arbitrarily, one or more of the μ's might be infinite. In the present case the μ's are numerical multiples of the a's †; and, consequently, the μ's are finite whenever the a's are finite. E. g. $(n = r = 2)$,

$$\mathfrak{f}_1(x, a) = x_1 + a_2,$$
$$\mathfrak{f}_2(x, a) = e^{a_2} x_2 + a_1 \left(\frac{e^{a_2} - 1}{a_2}\right),$$

$$a_1 = \Phi_1(\mu, \bar{a}) \equiv \frac{\bar{a}_1 (e^{\bar{a}_2} - 1)(\bar{a}_2 - \mu_2) e^{\bar{a}_2 - \mu_2}}{\bar{a}_2 e^{\bar{a}_2} (e^{\bar{a}_2 - \mu_2} - 1)} + \frac{\mu_1 (\bar{a}_2 - \mu_2)}{\mu_2 e^{\mu_2} (e^{\bar{a}_2 - \mu_2} - 1)} - \frac{\mu_1 (\bar{a}_2 - \mu_2)}{\mu_2 (e^{\bar{a}_2 - \mu_2} - 1)},$$

$$a_2 = \Phi_2(\mu, \bar{a}) \equiv \bar{a}_2 - \mu_2,$$

which give

$$a_1 = \Phi_1(\mu, a^{(0)}) \equiv -\mu_1,$$
$$a_2 = \Phi_2(\mu, a^{(0)}) \equiv -\mu_2.$$

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* In the present case the values of the a's giving the identical transformation are $a_1^{(0)} = a_2^{(0)} = 0$.

† Cf. Lie: Transformationsgruppen, III. 607 *et seq.*

Therefore, the $\bar{a}$'s and a's being taken arbitrarily, we have

$$\mathfrak{T}_{\bar{a}}\,\mathfrak{T}_{a} = \mathfrak{T}_{a},$$

where

$$a_k = \Phi_k\left[\mathfrak{M}\,(a,\, a^{(0)}),\, \bar{a}\right] \quad (k = 1,\, 2 \ldots r).$$

But, whereas the parameters a with which we dealt in that part of this paper, pages 240–248, relating to the first fundamental theorem were not restricted in range, the present parameters a, involved in the transformations $\mathfrak{T}_a$ defined by equations (A), must be finite, at least if the transformation $\mathfrak{T}_a$ is to be generated by an infinitesimal transformation. And if a and $\bar{a}$ are chosen arbitrarily,

$$a_k = \Phi_k\,(a,\, \bar{a}) \quad (k = 1,\, 2 \ldots r)$$

may be infinite.

Thus, in the example chosen, if $\bar{a}_1 \neq 0$, $a_1 \neq 0$, and $\bar{a}_2 + a_2 =$ an even multiple of πi, the transformation

$$x_1{}' = \mathfrak{f}_1\,(x,\, a) \equiv x_1 + a_2$$

$$x_2{}' = \mathfrak{f}_2\,(x,\, a) \equiv e^{a_2}x_2 + a_1\left(\frac{e^{a_2} - 1}{a_2}\right)$$

is finite, while

$$a_1 = \Phi_1\,(a,\, \bar{a}) \equiv \frac{a_2 + \bar{a}_2}{e^{a_2+\bar{a}_2} - 1}\left[\frac{\bar{a}_1}{\bar{a}_2}\,(e^{\bar{a}_2} - 1) + e^{\bar{a}_2}\frac{a_1}{a_2}(e^{a_2} - 1)\right]$$

is infinite; and the transformation $\mathfrak{T}_a$ cannot be generated by an infinitesimal transformation.